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ON REISSNER'S THEORY OF THE BENDING OF PLATES

By A. L. Goldenveizer

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ON REISSNER'S THEORY OF THE BENDING OF PLATES*

By A. L. Goldenveizer

The classical theory of the bending of thin plates is affected with a formal contradiction as is well known. The contradiction consists in the noncorrespondence of the order of the differential equations with the number of boundary conditions. In order to remove this contradiction a method for the relaying of static boundary conditions is used, e.g. the given boundary reactions are replaced by others which are statically equivalent on any section of the boundary.

In 1944 E. Reissner in article [1] proposed a new theory of the bending of thin plates, in which the order of the equations agrees with the number of boundary conditions. This article, persuasive in its clearness and elegance, called forth a widespread reaction in scientific literature, and is the main subject discussed in the present work.

1. In the article [1] E. Reissner considers a plate of constant thickness, acted upon by normal forces of variable intensity $(1/2)p$ and $-(1/2)p$, applied to the upper and lower boundary planes. It is assumed that forces of mass are absent.

The law of the stress distribution through the thickness of the plate is given by the equalities

$$\sigma_x = \frac{M_x}{h^2/6}\,\frac{z}{h/2} \qquad \sigma_y = \frac{M_y}{h^2/6}\,\frac{z}{h/2} \qquad \tau_{xy} = \frac{H}{h^2/6}\,\frac{z}{h/2} \tag{1.1}$$

where the axes x and y are considered to lie in the middle plane of the plate, and where h denotes the full thickness of the plate ($h=\text{const.}$).

The law of the distribution of the remaining stresses is determined from the equilibrium equations of the three-dimensional problem in the theory of elasticity,

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NASA reviewer's note: Several obviously typographical errors in equations in the original Russian text have been corrected by the reviewer without comment.

$$\tau_{xz} = \frac{V_x}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \qquad \tau_{yz} = \frac{V_y}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right]$$

$$\sigma_z = -\frac{3}{4}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right)\left[\frac{z}{h/2} - \frac{1}{3}\left(\frac{z}{h/2}\right)^3\right] \tag{1.2}$$

where

$$V_x = \frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} \qquad V_y = \frac{\partial H}{\partial x} + \frac{\partial M_y}{\partial y} \tag{1.3}$$

It is easy to verify that if the relation

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + p = 0 \tag{1.4}$$

holds, then all three equilibrium equations in the general problem of the theory of elasticity will be satisfied (in the absence of forces of mass), as well as the boundary conditions

$$\text{for} \quad z = \pm \frac{h}{2} \qquad \tau_{xz} = \tau_{yz} = 0 \qquad \sigma_z = \pm \frac{1}{2} p \tag{1.5}$$

which correspond to the problem formulated above.

The equalities

$$\int_{-h/2}^{+h/2} z\sigma_x \, dz = M_x \qquad \int_{-h/2}^{+h/2} z\sigma_y \, dz = M_y \qquad \int_{-h/2}^{+h/2} z\tau_{xy} \, dz = H$$

$$\int_{-h/2}^{+h/2} \tau_{xz} \, dz = V_x \qquad \int_{-h/2}^{+h/2} \tau_{yz} \, dz = V_y \tag{1.6}$$

hold, from which it follows that M_x, M_y, H, V_x, V_y are to be understood as moments and shearing stress resultants in the classical plate theory. Thus, the equalities (1.3) and (1.4) are the heart of the equilibrium equation, which must be satisfied by the moments and the shearing stresses. These, as is well known, can be reduced to the single equation

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 H}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + p = 0 \tag{1.7}$$

We introduce into the discussion the stress energy of the plate, defined by the following formula from the theory of elasticity

$$\Pi = \frac{1}{2E}\iiint\left\{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu\left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z\right) + 2(1+\nu)\left(\tau_x^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\right\}dx\,dy\,dz \tag{1.8}$$

By virtue of (1.1) and (1.2) this equality is brought to the form

$$\Pi = \frac{1}{2E}\iint\left\{\frac{12}{h^3}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H^2\right] + \frac{12(1+\nu)}{5h}\left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x}\right)^2\right] - \frac{12\nu}{5h}\,p\left(M_x + M_y\right) + \int_{-h2}^{+h/2}\sigma_z^2\,dz\right\}dx\,dy \tag{1.9}$$

Further, E. Reissner applies Castigliano's principle, which states that of all the statically possible stressed states, that state arises in the elastic body for which Π is a minimum.

We get a problem for the stipulated extremum (M_x, M_y, H, V_x, V_y must satisfy the equilibrium equations), for the solution of which E. Reissner proposes to apply undetermined Lagrange multipliers. Thus, for instance, it is possible in (1.9) to express V_x, V_y in terms of M_x, M_y, H, with the help of (1.3), and then add to the subintegral expression in formula (1.9) the summand

$$2Ehw\left[\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 H}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + p\right] \tag{1.10}$$

where w is a Lagrange multiplier, coinciding in meaning with the normal bend in the classical theory of plates, if by E we understand Young's modulus. Having carried out a variation with respect to the variables M_x, M_y, H, E. Reissner obtained the equalities

$$M_x - \frac{h^2}{5(1 - \nu)}\left(\frac{\partial V_x}{\partial x} + \nu \frac{\partial V_y}{\partial y}\right) - \frac{h^2 \nu}{10(1 - \nu)} p = - D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y - \frac{h^2}{5(1 - \nu)}\left(\frac{\partial V_y}{\partial y} + \nu \frac{\partial V_x}{\partial x}\right) - \frac{h^2 \nu}{10(1 - \nu)} p = - D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \qquad (1.11)$$

$$H - \frac{h^2}{10}\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right) = -(1 - \nu) D \frac{\partial^2 w}{\partial x \, \partial y}$$

(where D is the cylindrical rigidity), which must be satisfied at all points of the middle of the plate, and the equality

$$M_n\left[\frac{\partial w}{\partial n} - \frac{12(1 + \nu)}{5hE} V_n\right] + H_{ns}\left[\frac{\partial w}{\partial s} - \frac{12(1 + \nu)}{5hE} V_s\right] + V_n w = 0 \qquad (1.12)$$

which must be fulfilled on the boundary.

It is obvious that Equations (1.1) are a generalization of the elasticity relations of the classical plate theory, and differ from these only by those summands of the left members of the equation which contain the multiplier h^2.

The equality (1.12) determines the formulation of the boundary conditions. In the theory described, the static boundary conditions have the form

$$M_n = \overline{M}_n \qquad H_{ns} = \overline{H}_{ns} \qquad V_n = \overline{V}_n \qquad (1.13)$$

and the geometric (homogeneous) boundary conditions are written thus:

$$\frac{\partial w}{\partial n} = \frac{12(1 + \nu)}{5hE} V_n \qquad \frac{\partial w}{\partial s} = \frac{12(1 + \nu)}{5hE} V_s \qquad w = 0 \qquad (1.14)$$

We have, in this manner, three boundary conditions, on each boundary which is the main difference between the theory presented by E. Reissner, and the classical theory of thin plates.

2. The results presented in Sec. 1 permit of an obvious generalization. Instead of (1.1) and (1.2) we can write

$$\sigma_x = M_x\varphi(z) \qquad \sigma_y = M_y\varphi(z) \qquad \tau_{xy} = H\varphi(z)$$

$$\tau_{xz} = V_x\Phi)z) \qquad \tau_{yz} = V_y\Phi(z) \qquad \sigma_z = p\psi(z) \tag{2.1}$$

where

$$\Phi(z) = \int_z^{-h/2} \varphi(z)dz \qquad \psi(z) = \int_0^z \Phi(z)dz$$

Here $\varphi(z)$ is an odd function satisfying the conditions

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$$\int_{-h/2}^{+h/2} z\varphi(z)dz = 1 \qquad \psi\left(\frac{h}{2}\right) = \psi\left(-\frac{h}{2}\right) = \int_0^{h/2} \Phi(z)dz = \frac{1}{2} \tag{2.2}$$

The generalization consists in having the linear law of the stress distribution through the thickness of the plate replaced by an arbitrary inverse symmetrical law which obeys only the integral conditions (2.2). For

$$\varphi(z) = \frac{12}{h^3} z \tag{2.3}$$

we shall go back to the results of Sec. 1.

It is easy to verify that if M_x, M_y, H, V_x, V_y satisfy Equations (1.3) and (1.4) then all three equilibrium equations of the general elasticity theory and the boundary conditions (1.5) will be satisfied. The relations (1.6) remain valid as a consequence of (2.2) and consequently, as before, M_x, M_y, H, V_x, V_y can be considered as moments and stresses, and (1.3) and (1.4) as equilibrium equations which they must satisfy.

For the stress energy, instead of (1.9), the formula

$$\Pi = \frac{1}{2E} \iint \left\{ A\left[M_x^2 + M_y^2 - 2\nu M_xM_y + 2(1-\nu)H^2\right] + 2(1+\nu)B\left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x}\right)^2\right] - 2\nu Cp\left(M_x + M_y\right) + \int_{-h/2}^{+h/2} \sigma_z^2\, dz \right\} dx\, dy \tag{2.4}$$

will hold, where

$$A = \int_{-h/2}^{+h/2} \varphi^2(z)dz, \qquad B = \int_{-h/2}^{+h/2} \Phi^2(z)dz, \qquad C = \int_{-h/2}^{+h/2} \varphi(z)\psi(z)dz$$

Application of Castigliano's principle brings us to the relations

$$M_x - \frac{2B}{A(1-\nu)}\left(\frac{\partial V_x}{\partial x} + \nu\frac{\partial V_y}{\partial y}\right) - \frac{\nu C}{A(1-\nu)}\, p = -\frac{E}{A(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y - \frac{2B}{A(1-\nu)}\left(\frac{\partial V_y}{\partial y} + \nu\frac{\partial V_x}{\partial x}\right) - \frac{\nu C}{A(1-\nu)}\, p = -\frac{E}{A(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad (2.5)$$

$$H - \frac{B}{A}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) = -\frac{E}{A(1+\nu)}\frac{\partial^2 w}{\partial x\,\partial y}$$

Here E owes its origin to the Lagrange multiplier (1.10). It need not necessarily be identified with Young's modulus. E can be chosen in such a way that the equality

$$\frac{E}{A(1-\nu^2)} = D$$

is satisfied. Then the right members of (2.5) will coincide with the right members of (1.11).

With regard to the magnitudes B/A and C/A, these depend on the choice of the functions $\varphi(z)$. This means that into the summands by which E. Reissner's theory differs from the classical theory, there enter numerical coefficients depending on our arbitrary choice.

The numerical coefficients may theoretically oscillate between unbounded limits, as follows from the following example.

Let

$$\varphi(z) = -\frac{1}{h\epsilon} \quad \text{for} \quad -h/2 < z < -h/2 + \epsilon \quad \text{and for} \quad h/2 - \epsilon < z < h/2$$

$$\varphi(z) = 0 \quad \text{for} \quad -h/2 + \epsilon < z < +h/2 - \epsilon$$

(Such a function is used by E. Reissner in [2] for the formulation of the theory of sandwich-plates.) Then

$$\lim_{\epsilon\to 0}\int_{-h/2}^{+h/2} z\varphi(z)dz = 1, \qquad \lim_{\epsilon\to 0} A = \lim_{\epsilon\to 0}\int_{-h/2}^{+h/2}\varphi^2 dz = \lim_{\epsilon\to 0}\frac{2}{h^2\epsilon}$$

$$\lim_{\epsilon\to 0}\Phi(z) = \frac{1}{h} \qquad \lim_{\epsilon\to 0} B = \lim_{\epsilon\to 0}\int_{-h/2}^{+h/2}\Phi^2 dz = \frac{1}{h}$$

$$\lim_{\epsilon \to 0} \Psi(z) = \frac{z}{h} \qquad \lim_{\epsilon \to 0} C = \lim_{\epsilon \to 0} \int_{-h/2}^{+h/2} \varphi(z)\Psi(z)dz = \frac{1}{h}$$

If we choose φ according to formula (2.3), then

$$A = \frac{12}{h^3} \qquad B = \frac{6}{5h} \qquad C = \frac{5}{5h}$$

B and C have changed little, but A can be increased infinitely. Common sense and innumerable computational data tell us that formula (2.3) reflects approximately correctly the true law of the distribution of $\sigma_x, \sigma_y, \tau_{xy}$ away from boundaries and other lines (or points) of deformation of the strained state. However, in boundary zones formula (2.3) can turn out to be very far from reality. The question arises as to what exerts the greatest influence on the correction contributed by E. Reissner's theory: the elastic phenomena occurring near the boundary, or the elastic phenomena originating away from it? Let us consider an example.

3. In article [2] E. Reissner has shown, that according to the plate theory proposed by him, the moment M_r, $H_{r\theta}$ and the shearing stresses V_r, V_θ, in polar coordinates, in the absence of surface load, are expressed by formulas

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] + 2k^2 \frac{\partial V_r}{\partial r}$$

$$H_{r\theta} = -(1 - \nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) + k^2\left[\frac{1}{r}\frac{\partial V_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right] \qquad (3.1)$$

$$V_r = \frac{1}{r}\frac{\partial \chi}{\partial \theta} \qquad V_\theta = -\frac{\partial \chi}{\partial r}$$

Here k^2 is a small parameter commensurable with h, whose exact meaning depends on the choice of $\varphi(z)$; if the latter is determined by formula (2.3), then

$$k^2 = \frac{1 - \nu}{2}\frac{D}{G}$$

where G is the shear modulus. The bending of the plate w, satisfies the equation

$$D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]w=\varphi$$

where φ is the real part of the analytic function $\varphi + i\psi$, of complex argument $re^{i\theta}$. The function $\chi = \psi_1 - \psi$, where ψ_1 satisfies the equation

$$\psi_1 - k^2\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\psi_1 = 0$$

We note that when $k = 0$, we shall have $\psi_1 = 0$, i.e. we return to the classical plate theory. We shall consider the computation problem of a circular plate, without load on the surface, to whose free boundary are applied the following forces: the bending moment $m_r \cos n\theta$, the transverse force $\vartheta_r \cos n\theta$, the moment of rotation $h_{r\theta} \sin n\theta$. (Here m_r, ϑ_r, $h_{r\theta}$ are constant, $n \geq 2$).

The system of boundary reactions is in self equilibrium, and a solution can be sought in the form:

$$\varphi = \psi^*(r)\cos n\theta \qquad \psi = \psi^*(r)\sin n\theta \qquad Dw = w^*(r)\cos n\theta \qquad \psi_1 = \psi_1^*(r)\sin n\theta$$

where the starred magnitudes must satisfy the limiting conditions where $r = 0$. Therefore we can let

$$\varphi^* = \varphi^* = A_1\frac{r^n}{r_0^n} \qquad w^* = \left(\frac{1}{4}\frac{A_1}{n+1}\frac{r^{n+2}}{r_0^n} + A_2\frac{r^n}{r_0^{n-2}}\right)$$

where r_0 is the radius of the plate.

The function ψ_1^* must satisfy the equation

$$\psi_1^* - k^2\left(\frac{d^2\psi_1^*}{dr^2}+\frac{1}{r}\frac{d\psi_1^*}{dr}-\frac{n^2}{r^2}\psi_1^*\right) = 0$$

and the limiting condition, when $r = 0$. We shall set, therefore,

$$\psi_1^* = A_3 \sqrt{\frac{2\pi r_0}{k}} \exp\left(-\frac{r_0}{k}\right) J_n\left(\frac{r_0}{k}\right)$$

where J_n is a modified Bessel function of the first type [3], which is defined for integral n by the following series

$$J_n(t) = \sum_{s=0}^{\infty} \frac{1}{s!(n+s)!}\left(\frac{t}{2}\right)^{n+2s}$$

The asymptotic expansion of J_n for large t's has the form[3]

$$J_n(t) = \frac{1}{\sqrt{2\pi t}} e^t \left\{1 - \frac{4n^2 - 1}{1!8t} + \frac{(4n^2 - 1)(4n^2 - 3^2)}{2!(8t)^2} - \dots\right\}$$

$$+ \exp\left[i(n + 1/2)\pi\right] \frac{1}{\sqrt{2\pi t}} e^{-t} \left\{1 + \frac{4n^2 - 1}{1!8t} + \frac{(4n^2 - 1)(4n^2 - 3^2)}{2!(8t)^2} + \dots\right\}$$

so that we can let

$$J_n(t) \approx \frac{1}{\sqrt{2\pi t}} e^t \left(1 - \frac{4n^2 - 1}{8t}\right)$$

In addition the formulas [3]

$$J_n{}' = \frac{1}{2}\left(J_{n-1} + J_{n+1}\right) \qquad J_n{}'' = \frac{1}{4}\left(J_{n-2} + 2J_n + J_{n+2}\right)$$

hold, from which

$$J_n{}^t(t) \approx \frac{1}{\sqrt{2\pi t}} e^t \left[1 - \frac{4(n^2 + 1) - 1}{8t}\right] \qquad J_n{}''(t) \approx \frac{1}{\sqrt{2\pi t}} e^t \left[1 - \frac{4(n^2 + 3) - 1}{8t}\right]$$

On the boundary of the plate, t equal to $k^{-1}r_0$ is very large, and applying the derived formulas (the approximate equality signs are replaced by exact equality signs) we shall get

$$\psi_1^*\Big|_{r=r_0} = A_3\left[1 - \frac{4n^2 - 1}{8r_0}\,k\right]$$

$$\frac{d\psi_1^*}{dr}\Bigg|_{r=r_0} = \frac{1}{k}\,A_3\left[1 - \frac{4(n^2+1) - 1}{8r_0}\,k\right] \tag{3.2}$$

$$\frac{d^2\psi_1^*}{dr^2}\Bigg|_{r=r_0} = \frac{1}{k^2}\,A_3\left[1 - \frac{4(n^2+3) - 1}{8r_0}\,k\right]$$

Substituting in E. Reissner's computation formulas the expressions for w and χ and disregarding magnitudes of order k^2, after a comparison with magnitudes of order one, we shall find

$$M_r = -\left[\frac{n + 2 - \nu(n-2)}{4}\,A_1\,\frac{r^n}{r_0^n} + n(n-1)(1-\nu)A_2\,\frac{r^{n-2}}{r_0^{n-2}} + \frac{2k^2 n}{r}\left(\frac{\psi_1^*}{r} - \frac{d\psi_1^*}{dr}\right)\right]\cos n\theta$$

$$H_{r\theta} = \left[\frac{n(1-\nu)}{4}\,A_1\,\frac{r^n}{r_0^n} + n(n-1)(1-\nu)A_2\,\frac{r^{n-2}}{r_0^{n-2}} - 2k^2\,\frac{d^2\psi_1^*}{dr^2} + \psi_1^*\right]\sin n\theta$$

$$V_r = \frac{1}{r_0}\left[nA_1\,\frac{r^{n-1}}{r_0^{n-1}} - \frac{r_0 n}{r}\,\psi_1^*\right]\cos n\theta$$

For the determination of the constants of integration, we get, with the aid of the boundary conditions, equations of the form

$$-\frac{n + 2 - \nu(n-2)}{4}\,A_1 - n(n-1)(1-\nu)A_2 - \frac{2nk}{r_0}\left[1 - \frac{4n^2 + 7}{8r_0}\,k\right]A_3 = m_r$$

$$\frac{n(1-\nu)}{4}\,A_1 + n(n-1)(1-\nu)A_2 - \left[1 - \frac{4(n^2+6) - 1}{8r_0}\,k\right]A_3 = h_{r\theta} \tag{3.3}$$

$$\frac{n}{r_0}\,A_1 - \frac{n}{r_0}\left[1 - \frac{4n^2 - 1}{8r_0}\,k\right]A_3 = v_2$$

We multiply the second of these equations by $-n/r_0$, add it to the third equation, and replace the second equation by the combination obtained

$$-\frac{n+2-\nu(n-2)}{4}A_1 - n(n-1)(1-\nu)A_2 - \frac{2nk}{r_0}\left[1 - \frac{4n^2+7}{8}\frac{k}{r_0}\right]A_3 = m_r$$

$$\frac{n}{r_0}\left[1 - \frac{n(1-\nu)}{4}\right]A_1 - \frac{n(n-1)(1-\nu)}{r_0}A_2 - \frac{n}{r_0}\frac{25}{8}\frac{k}{r_0}A_3 = \upsilon_r - \frac{nh_{r\theta}}{r_0}$$

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$$\frac{n}{r_0}A_1 - \frac{n}{r_0}\left[1 - \frac{4n^2-1}{8r_0}k\right]A_3 = \upsilon_r$$

The solution of the system obtained has the form

$$A_i = A_{i0} + \frac{k}{r_0}A_{i1} + \ldots \qquad (i = 1, 2, 3)$$

where A_{i0} is the solution of the "boundary" system obtained from (3.3) where $k = 0$, and A_{i1} is the first correction, which can be obtained, for instance, by the method of successive approximations.

The boundary system is broken up, A_{10} and A_{20} are determined from the first two equations of this system, which do not differ from the equations obtained for the computation of the plate in accordance with the classical theory (relaxed static boundary conditions).

Thus E. Reissner's theory makes it possible : a) to introduce corrections of order k after a comparison with unity in the constants A_1, A_2, corresponding to the classical plate theory; b) to determine the new constant A_3 and construct an additional stress state related to the function ψ_1.

4. The asymptotic formulas (3.2) show that $J_n(t)$ decreases quickly with t for large t's. Consequently the stress state related to the constant A_3 is of pronounced local character. It is damped as it becomes remote from the boundary, as $\exp(-h^* s)$, where $h^* = h/r_0\sqrt{k}$, s is the distance from the boundary along the normal, i.e. it is faster than the simple boundary effect in the edge region which is damped according to the law $\exp\left(-\sqrt{h^* s}\right)$.

This stressed state is called by E. Reissner the boundary effect: it differs, however, qualitatively from the latter, and deserves a special name. We shall call it, from now on, "Reissner's boundary effect".

We shall let in the example considered (sec. 3)

$$m_r = \upsilon_r - \frac{nh_{r\theta}}{r_0} = 0 \qquad \upsilon_r \neq 0$$

(the system of boundary reactions is statistically equivalent to zero on any section of the boundary).

Then the system (3.3) will give

$$A_{10} = A_{20} = 0 \qquad A_{30} \approx \upsilon_r \qquad A_{11},\ A_{21},\ A_{31} \sim \upsilon_r$$

The stress state determined by the classical theory vanishes. What remain are: a) Reissner's boundary effect, and b) a "correctional" stress state whose intensity is proportional to $k/r_0 \sim h^*$.

Thus two systems of boundary forces are statically equivalent to each other, and evoke in the plate elastic phenomena, differing by the stress state, the intensity of which falls extremely quickly while becoming remote from the boundary, from magnitudes of order υ_r up to magnitudes of $h^* \upsilon_r$.

Turning our attention to the first two of the computing formulas (3.1), we note that the terms which correct the classical theory contain the multiplier k^2. These have a value in the immediate proximity of the boundary, i.e. when it is still necessary to take into consideration the quickly damped function ψ_1^*, the derivatives of which are large compared to the function itself (formulas 3.2).

Consequently the corrections of E. Reissner's theory are reduced to the above described Reissner's boundary effect and the "correctional" stress state. The intensity of both these stress states is related to elastic phenomena taking place in the immediate proximity of the boundary, where the law of the distribution of the stresses accepted by Reissner can turn out to be far from reality.

This means that E. Reissner's theory, while giving a correct qualitative picture of a phenomenon, may also give false magnitudes for the corrections. The shortcoming of this theory is that it is based on hypotheses which reflect phenomena occurring away from the boundary of the plate, while the main part is played (or at any rate may be played) by phenomena holding near to the boundaries.

5. In conclusion we will touch on the work† of B. F. Vlasov [4]. In this article the same problem as in article [1] is considered. The law of the variation of displacements on the thickness of the plate is given.

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† This paragraph was written by the author in the light of the appearance of article [4], after the main part of the article had already been submitted for print.

$$u(x,y,z) = -zt_x - \frac{4z^3}{3h^2}\frac{1}{G}\tau_{xz}{}^{o}$$

$$\upsilon(x,y,z) = -zt_y - \frac{4z^3}{3h^2}\frac{1}{G}\tau_{yz}{}^{o} \tag{5.1}$$

$$w = w(x,y)$$

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$$\tau_{xz}{}^{o} = G\left(\frac{\partial w}{\partial x} - t_x\right) \qquad \tau_{yz}{}^{o} = G\left(\frac{\partial w}{\partial y} - t_y\right) \tag{5.2}$$

This comprises one of the two assumptions which are at the basis of B. F. Vlasov's theory. From here with the aid of Hooke's law the tensions are computed, and then the moments and shearing stresses. We have (in Reissner's notations)

$$M_x = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial t_x}{\partial x} + \nu\frac{\partial t_y}{\partial y}\right) - \frac{Eh^3}{60(1-\nu^2)}\frac{1}{G}\left(\frac{\partial\tau_{xz}{}^{o}}{\partial x} + \nu\frac{\partial\tau_{yz}{}^{o}}{\partial y}\right)$$

$$M_y = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial t_y}{\partial y} + \nu\frac{\partial t_x}{\partial x}\right) - \frac{Eh^3}{60(1-\nu^2)}\frac{1}{G}\left(\frac{\partial\tau_{yz}{}^{o}}{\partial y} + \nu\frac{\partial\tau_{xz}{}^{o}}{\partial x}\right) \tag{5.3}$$

$$H = -\frac{Eh^3}{24(1-\nu^2)}\left(\frac{\partial t_x}{\partial y} + \frac{\partial t_y}{\partial x}\right) - \frac{Eh^3}{120(1+\nu)}\frac{1}{G}\left(\frac{\partial\tau_{xz}{}^{o}}{\partial y} + \frac{\partial\tau_{yz}{}^{o}}{\partial x}\right)$$

$$V_x = \frac{2}{3}h\tau_{xz}{}^{o} \qquad V_y = \frac{2}{3}h\tau_{yz}{}^{o}$$

Thus, the equations of the continuity of deformations and Hooke's law of the three dimensional problem of the theory of elasticity are fulfilled exactly. The equilibrium equations are satisfied integrally, with the aid of Lagrange's principle, using not the exact expression for the variation of energy, but the approximate formula

$$\delta U = \iint\left\{-M_x\delta\left(\frac{\partial t_x}{\partial x}\right) - M_y\delta\left(\frac{\partial t_y}{\partial y}\right) - H\delta\left(\frac{\partial t_x}{\partial y} + \frac{\partial t_y}{\partial x}\right)\right.$$

$$\left. + V_x\delta\left(\frac{\partial w}{\partial x} - t_x\right) + V_y\delta\left(\frac{\partial w}{\partial y} - t_y\right)\right\} dx\, dy \tag{5.4}$$

After the usual transformations, this formula brings us to the equilibrium equations of the classical plate theory (1.3), (1.4), and to the formulation of the boundary conditions, which differ from those of Reissner only by the value of the numerical multipliers in the coefficients for V_n and V_s in the formulas (1.14).

B. F. Vlasov asserts that formula (5.4) follows from the fact that in the theory of plates the problem of the determination of stresses is replaced by the problem of moments and stresses. (This fact is considered by B. F. Vlasov as the second fundamental assumption in his work). We cannot agree with such a proposition. For instance, the energy variation of the stresses σ_x will be expressed thus F 27

$$\iint \left\{ \int_{-h/2}^{+h/2} \sigma_x \delta\left(\frac{\partial u}{\partial x}\right) dz \right\} dx\ dy \tag{5.5}$$

In addition

$$\int_{-h/2}^{+h/2} z\sigma_x\ dz = M_x \tag{5.6}$$

is given.

The question is posed as to how there can be obtained from (5.5) with the aid of (5.6) a term corresponding to the first summand of the subintegral expression (5.4). The answer is obvious. The "assumptions" (5.6) are insufficient. We must replace the first expression

$$u(x,y,z) = -zt_x - \frac{4z^2}{3h^2} \cdot \frac{1}{G} \tau^o{}_{xz} \tag{5.7}$$

by a simpler one.

$$u(x,y,z) = -zt_x \tag{5.8}$$

Geometrically, this denotes accepting the hypothesis of rectilinearity of the deformed normal. Thus formula (5.4) is not a consequence of B. F. Vlasov's second assumption. Besides it contradicts his first assumption, which states: "A rectilinear element of a plate which is normal to the middle plane until deformation, bends in the process of deformation in such a way, that displacements on the thickness of the plate vary parabolically.

From an analytical point of view replacing (5.7) by the relation (5.8) is equivalent to making the assumption that magnitudes of the form τ^{o}_{xz}/G are negligibly small by comparison with magnitudes of the form t_x. Meanwhile it is not difficult to see that the formulas (5.2), (5.3) proposed by B. F. Vlasov, differ from the formulas of the classical plate theory by terms of the same order.

Therefore the refinements of B. F. Vlasov are altogether devoid of authenticity.

We juxtapose articles [4] and [1].

Reissner:

given are: a) the law of variation of stresses σ_x, σ_y, τ_{xy} through the thickness; b) the equilibrium conditions and Hooke's law are exactly satisfied. (E. Reissner uses Hooke's law in the construction of the expression for the energy of tensions.) c) the continuity conditions for the deformations are integrally satisfied (with the aid of a correct application of Castigliano's principle.)

B. F. Vlasov:

a) given is the law of variation of u, v, w through the thickness; b) the continuity conditions for the deformations and Hooke's law are exactly satisfied (with the aid of an incorrect application of Lagrange's principle).

The law of change of u, v, w of B. F. Vlasov is in fact equivalent to the law of change of σ_x, σ_y, τ_{xy} of E. Reissner, since both lead to the same (parabolic) law of the distribution of displaced tensions.

The question naturally arises as to what are the advantages of article [4].

B. F. Vlasov considers it to be his special merit that he does not use the proposition that " . . .an element, initially rectilinear and normal to the middle plane, remains rectilinear even after deformation." It is difficult to understand why B. F. Vlasov considers the proposition about the rectilinearity of the deformed normal to be so vicious, but from the recapitulation of article [1] (para. 1) we can see that E. Reissner did not use this proposition either. Probably B. F. Vlasov wants to say that E. Reissner based himself somewhere on this proposition implicitly after all. This may not be devoid of foundation, but as we have seen, the same may be said of B. F. Vlasov's work.

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